Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 14 – Solutions

Exercise 1 (*Properties of line integrals*)**:** Let M be a smooth manifold with or without boundary. Let $\gamma\colon [a,b] \to M$ be a piecewise smooth curve segment in M, and let $\omega,\, \omega_1,\, \omega_2 \in \mathfrak{X}^*(M)$. Prove the following assertions:

(a) For any $c_1, c_2 \in \mathbb{R}$ we have

$$\int_\gamma \big(c_1\omega_1 + c_2\omega_2\big) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

(b) If γ is a constant map, then

$$\int_\gamma \omega = 0.$$

(c) If $\gamma_1 \coloneqq \gamma|_{[a,c]}$ and $\gamma_2 \coloneqq \gamma|_{[c,b]}$, where $a, b, c \in \mathbb{R}$ with $a < c < b$, then

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

(d) If $F\colon M \to N$ is any smooth map and if $\eta \in \mathfrak{X}^*(N)$, then

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Solution:

(a) Follows immediately from the corresponding property of usual integrals.

(b) Since γ is constant, for any $p \in [a,b]$ we have $d\gamma_p = 0$, and thus

$$(\gamma^*\omega)_p\,(v) = \omega_{\gamma(p)}\big(d\gamma_p(v)\big) = 0 \ \text{ for any } v \in T_p\,[a,b],$$

which implies that $\gamma^*\omega = 0$. Therefore,

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega = 0.$$

(c) Follows immediately from the corresponding property of usual integrals.

(d) By *Remark 8.17* we deduce that

$$\int_{\gamma} F^*\eta = \int_{[a,b]} \gamma^*\big(F^*\eta\big) = \int_{[a,b]} (F\circ\gamma)^*\eta = \int_{F\circ\gamma} \eta.$$

Exercise 2 (*Parameter independence of line integrals*)**:** Let M be a smooth manifold with or without boundary, $\omega \in \mathfrak{X}^*(M)$, and let γ be a piecewise smooth curve segment in M. Show that for any reparametrization $\widetilde{\gamma}$ of γ we have

$$\int_{\widetilde{\gamma}} \omega = \begin{cases} \displaystyle\int_{\gamma} \omega & \text{if } \widetilde{\gamma} \text{ is a forward reparametrization,} \\ \displaystyle -\int_{\gamma} \omega & \text{if } \widetilde{\gamma} \text{ is a backward reparametrization.} \end{cases}$$

Solution: Let $\varphi\colon [c,d] \to [a,b]$ be the strictly monotone, piecewise smooth function such that $\widetilde{\gamma} = \gamma\circ\varphi$. Let $a = a_0 < a_1 < \cdots < a_k = b$ be a partition of $[a,b]$ such that both $\gamma|_{[a_{i-1},a_i]}$ and $\varphi^{-1}|_{[a_{i-1},a_i]}$ are smooth for all $1 \le i \le k$. Set $c_i = \varphi^{-1}(a_i)$ for each $1 \le i \le k$. Then $\widetilde{\gamma}|_{[c_{i-1},c_i]}$ is smooth for all $1 \le i \le k$ by construction. Hence, we have

$$\int_{\widetilde{\gamma}} \omega = \sum_{i=1}^{k} \int_{[c_{i-1},c_i]} \widetilde{\gamma}^*\omega = \sum_{i=1}^{k} \int_{[c_{i-1},c_i]} \varphi^*\gamma^*\omega.$$

Let $\varepsilon \in \{-1,1\}$ be equal to 1 if φ is increasing, and equal to -1 if φ is decreasing. Note that if φ is increasing (resp. decreasing), then for all i the restriction $\varphi|_{[c_{i-1},c_i]}\colon [c_{i-1},c_i] \to [a_{i-1},a_i]$ is an increasing (resp. decreasing) diffeomorphism. Hence, by *Lemma 11.6* we obtain

$$\int_{\widetilde{\gamma}} \omega = \sum_{i=1}^{k} \varepsilon \int_{[a_{i-1},a_i]} \gamma^*\omega = \varepsilon \int_{\gamma} \omega,$$

which is what we wanted to prove.

Exercise 3: Let M be a compact, connected, oriented, smooth n-manifold without boundary (i.e., $\partial M = \emptyset$), where $n \ge 1$, and let $\omega \in \Omega^{n-1}(M)$. Show that there exists a point $p \in M$ such that $(d\omega)_p = 0 \in \Lambda^n(T_p^*M)$.

Solution: Assume on the contrary that $d\omega \in \Omega^n(M)$ is an orientation form on M. Since M is connected, $d\omega$ must be either positively or negatively oriented, and hence

$$\int_M d\omega \neq 0$$

by *Proposition 11.20.* On the other hand, Stokes' theorem, together with the fact that $\partial M = \emptyset$, yield

$$\int_M d\omega = \int_{\partial M} \omega = 0.$$

This contradiction shows that there is a point $p \in M$ such that $(d\omega)_p = 0 \in \Lambda^n(T_p^*M)$.

Exercise 4:

(a) Let M be a smooth n-manifold (without boundary) and let $\omega \in \Omega^1(M)$.

(i) Let $\big(U,(x^i)\big)$ be a smooth coordinate chart for M, and write $\omega|_U = \sum_{i=1}^n \omega_i \, dx^i$ in this chart. Find an expression for the *exterior derivative* $d\omega \in \Omega^2(M)$ of ω in this chart (that is, an expression of $d\omega$ in terms of the natural basis induced in each fiber of $\Lambda^2(T^*M)$ by the given chart).

(ii) Deduce that ω is closed if and only if for every point $p \in M$ there exists a smooth coordinate chart $\big(U,(x^i)\big)$ such that $p \in U$ and

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j} \quad \text{for all } 1 \le i, j \le n,$$

where $\omega|_U = \sum_{i=1}^n \omega_i \, dx^i$ in this chart.

(b) Consider the smooth 1-forms

$$\omega = y\cos(xy)\,dx + x\cos(xy)\,dy \in \Omega^1(\mathbb{R}^2)$$

and

$$\eta = x\cos(xy)\,dx + y\cos(xy)\,dy \in \Omega^1(\mathbb{R}^2).$$

(i) Show that ω is closed and exact.

(ii) Show that η is neither closed nor exact.

(iii) Compute $\omega \wedge \eta$.

(c) Evaluate the line integral

$$\int_\gamma \omega,$$

where γ is the straight line segment from $(0,0)$ to $(\sqrt{\pi}, \sqrt{\pi})$.

Solution:

(a)(i) By definition of the exterior derivative, we have

$$d\omega = \sum_{i=1}^n d\omega_i \wedge dx^i,$$

and since

$$d\omega_i = \sum_{j=1}^n \frac{\partial \omega_i}{\partial \omega_j}\, dx^j \quad \text{for every } 1 \le i \le n,$$

we obtain

$$\begin{aligned} d\omega &= \sum_i \left(\sum_j \frac{\partial \omega_i}{\partial \omega_j} dx^j \right) \wedge dx^i = \sum_{i,j} \frac{\partial \omega_i}{\partial \omega_j}\, dx^j \wedge dx^i \\ &= \sum_{j<i} \frac{\partial \omega_i}{\partial \omega_j}\, dx^j \wedge dx^i + \sum_{j=i} \frac{\partial \omega_i}{\partial \omega_j} \underbrace{dx^j \wedge dx^i}_{=0} + \sum_{j>i} \frac{\partial \omega_i}{\partial \omega_j} \underbrace{dx^j \wedge dx^i}_{=-dx^i \wedge dx^j} \\ &= \sum_{j<i} \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i. \end{aligned}$$

(a)(ii) Assume first that ω is closed, and let $\big(U,(x^i)\big)$ be an arbitrary smooth chart for M. By the above computation we have

$$0 = d\omega|_U = \sum_{j<i} \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i,$$

and since $\{dx^j \wedge dx^i\}_{j<i}$ gives a basis in each fiber of $\Lambda^2(T^*M)$, we have

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$$

for all $j < i$. By symmetry, the equation holds in fact for all i, j, so we deduce the forward direction.

For the reverse direction, let $p \in M$ be arbitrary and let $\big(U,(x^i)\big)$ be a smooth chart around $p \in M$ such that on U we have

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j} \quad \text{for all } 1 \leq i, j \leq n.$$

By part (a)(i), we obtain

$$d\omega = \sum_{j<i} \underbrace{\left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right)}_{=0} dx^j \wedge dx^i = 0,$$

and thus $d\omega_p = 0$. As $p \in M$ was arbitrary, we conclude that $d\omega = 0$, so ω is closed.

(b)(i) Consider the function

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \;\; (x, y) \mapsto \sin(xy)$$

and observe that $df = \omega$; in other words, ω is exact. Hence, ω is closed. (This can also be verified with a direct computation).

(b)(ii) We have

$$\begin{aligned} d\eta &= d\big(x\cos(xy)\big) \wedge dx + d\big(y\cos(xy)\big) \wedge dy \\ &= \Big(\big(\cos(xy) - xy\sin(xy)\big)dx + \big(-x^2\sin(xy)\big)\,dy \Big) \wedge dx \, + \\ &\quad + \Big(-y^2\sin(xy)\,dx + \big(\cos(xy) - xy\sin(xy)\big)dy \Big) \wedge dy \\ &= -x^2\sin(xy)\,dy \wedge dx - y^2\sin(xy)\,dx \wedge dy \\ &= (x^2 - y^2)\sin(xy)\,dx \wedge dy, \end{aligned}$$

which does not vanish identically on $\mathbb{R}^2$. Therefore, η is not closed (see also part (a)(ii)), and thus η cannot be exact either.

(b)(iii) We compute that

$$\begin{aligned} \omega \wedge \eta &= \big(y\cos(xy)\,dx + x\cos(xy)\,dy\big) \wedge \big(x\cos(xy)\,dx + y\cos(xy)\,dy\big) \\ &= y^2\cos^2(xy)\,dx \wedge dy + x^2\cos^2(xy)\,dy \wedge dx \\ &= (y^2 - x^2)\cos^2(xy)\,dx \wedge dy. \end{aligned}$$

(c) The straight line segment from $(0,0)$ to $(\sqrt{\pi}, \sqrt{\pi})$ can be parametrized by the smooth curve segment

$$\gamma \colon \big[0, \sqrt{\pi}\big] \to \mathbb{R}^2, \ t \mapsto (t,t).$$

Since $\omega = df$ is exact, by the fundamental theorem of line integrals we obtain

$$\begin{aligned}\int_\gamma \omega = \int_\gamma df &= (f \circ \gamma)\big(\sqrt{\pi}\big) - (f \circ \gamma)(0) \\ &= \sin(\pi) - \sin(0) = 0.\end{aligned}$$

Exercise 5: Consider the covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^3)$ given by

$$\omega = e^{y^2}\, dx + 2xye^{y^2}\, dy - 2z\, dz.$$

(a) Verify by direct computation that ω is closed.

(b) Using the fact that $\omega \in \mathfrak{X}^*(\mathbb{R}^3)$ is exact on the star-shaped set $\mathbb{R}^3$ (which follows from *Poincaré's lemma*), find a *potential* for ω, i.e., a function $f \in C^\infty(\mathbb{R}^3)$ such that $\omega = df$.

(c) Compute the line integral of ω along the smooth curve segment $\gamma \colon [0,1] \to \mathbb{R}^3, \ t \mapsto (t, t^2, t^3)$.

Solution:

(a) We have

$$\begin{aligned} d\omega &= \big(2ye^{y^2}\big) dy \wedge dx + \Big(2ye^{y^2} dx + \big(2xe^{y^2} + 4xy^2e^{y^2}\big) dy\Big) \wedge dy - 2\, dz \wedge dz \\ &= -\big(2ye^{y^2}\big) dx \wedge dy + \big(2ye^{y^2}\big) dx \wedge dy \\ &= 0. \end{aligned}$$

(b) For $f \in C^\infty(\mathbb{R}^3)$ to be a potential for ω, it must satisfy

$$\frac{\partial f}{\partial x} = e^{y^2}, \quad \frac{\partial f}{\partial y} = 2xye^{y^2}, \quad \frac{\partial f}{\partial z} = -2z. \tag{1}$$

Holding y and z fixed and integrating the first equation of (1) with respect to x, we obtain

$$f(x,y,z) = \int e^{y^2}\, dx = xe^{y^2} + C_1(y,z),$$

where the "constant" of integration $C_1(y,z)$ may depend on the choice of (y,z). Now, the second equation of (1) implies

$$2xye^{y^2} = \frac{\partial}{\partial y}\Big(xe^{y^2} + C_1(y,z)\Big) = 2xye^{y^2} + \frac{\partial C_1}{\partial y}(y,z),$$

which forces $\frac{\partial C_1}{\partial y} = 0$, so C_1 is actually a function of z only. Finally, the third equation of (1) yields

$$-2z = \frac{\partial}{\partial z}\Big(xe^{y^2} + C_1(z)\Big) = \frac{dC_1}{dz}(z),$$

which implies that $C_1(z) = -z^2 + c$, where $c \in \mathbb{R}$ is an arbitrary constant. Hence, a potential function for ω is given by

$$f\colon \mathbb{R}^3 \to \mathbb{R},\ f(x,y,z) = xe^{y^2} - z^2.$$

Any other potential differs from this one by a constant.

(c) Since $\omega = df$ is exact, by the fundamental theorem of line integrals we obtain

$$\begin{aligned}\int_\gamma \omega = \int_\gamma df &= (f \circ \gamma)(1) - (f \circ \gamma)(0) \\ &= f(1,1,1) - f(0,0,0) = e - 1.\end{aligned}$$